



## Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at <http://about.jstor.org/participate-jstor/individuals/early-journal-content>.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact [support@jstor.org](mailto:support@jstor.org).

$\therefore \alpha\beta + \gamma\delta = 2b/a$ . Since  $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = m$ , we have  $(\alpha\beta + \gamma\delta) + (\alpha + \beta) + (\gamma + \delta) = m$ , or  $(\alpha\beta + \gamma\delta) + (\alpha + \beta)^2 = m$ , or  $2b/a + \frac{1}{4}a^2 = m$

II. Solution by G. B. M. ZERRE, A. M., Ph. D., Parsons, W. Va.

Place  $x^4 + ax^3 + (2b/a + a^2/4)x^2 + bx + c = (x^2 + kx + m)(x^2 + kx + n)$  where  $k$  is the sum of two of the roots. Since the sum of two roots equals the sum of the other two,  $k$  must be the same in both factors. Equating like powers of  $x$  we get  $2k = a$ ,  $k^2 + m + n = 2b/a + a^2/4$ ,  $k(m + n) = b$ ,  $mn = c$ .

$\therefore k = a/2$ ,  $m + n = 2b/a$ ,  $mn = c$ , etc. The roots are now easily found—

$$m = \frac{1}{a} [b \pm \sqrt{b^2 - a^2 c}], \quad n = \frac{1}{a} [\mp \sqrt{b^2 - a^2 c}].$$

$$\therefore x = -\frac{1}{2}[k \mp \sqrt{k^2 - 4m}] = -\frac{1}{2}\left(\frac{a}{2} \mp \sqrt{\frac{a^2}{4} - \frac{4}{b} [b \pm \sqrt{b^2 - a^2 c}]}\right).$$

$$x = -\frac{1}{2}[k \mp \sqrt{k^2 - 4n}] = -\frac{1}{2}\left(\frac{a}{2} \mp \sqrt{\frac{a^2}{4} - \frac{4}{a} [b \mp \sqrt{b^2 - a^2 c}]}\right).$$

Analogously solved by F. D. Posey.

III. Solution by A. H. HOLMES, Brunswick, Maine, and L. E. NEWCOMB, Los Gatos, California.

Transposing  $c$  and adding  $b^2/a^2$  to both sides, the roots of the equation are easily found to be

$$x_1 = -\frac{1}{2}\left[\frac{a}{2} + \sqrt{\frac{a^2}{4} - \frac{4}{a} [b + \sqrt{b^2 - 4ac}]}\right]$$

$$x_2 = -\frac{1}{2}\left[\frac{a}{2} - \sqrt{\frac{a^2}{4} - \frac{4}{a} [b + \sqrt{b^2 - 4ac}]}\right]$$

$$x_3 = -\frac{1}{2}\left[\frac{a}{2} + \sqrt{\frac{a^2}{4} - \frac{4}{a} [b - \sqrt{b^2 - 4ac}]}\right]$$

$$x_4 = -\frac{1}{2}\left[\frac{a}{2} - \sqrt{\frac{a^2}{4} - \frac{4}{a} [b - \sqrt{b^2 - 4ac}]}\right]$$

Evidently,  $x_1 + x_2 = x_3 + x_4 = -\frac{1}{2}a$ .

Also solved by Elmer Schuyler.

204. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

If  $\alpha$ ,  $\beta$ ,  $\gamma$  be the roots of the cubic equation  $x^3 + qx + r = 0$ , prove that  $3\sum x^2 \sum x^5 = 5\sum x^3 \sum x^4$ .

## I. Solution by PHILIP GRABER, Ph. B., M. S., Akron, Ohio.

Since  $\alpha + \beta + \gamma = 0$  we may write  $\alpha = -2s$ ,  $\beta = s + t$ ,  $\gamma = s - t$ . Then

$$\Sigma \alpha^2 = 6s^2 + 2t^2 \dots (1), \quad \Sigma \alpha^3 = -6s^3 + 6st^2 \dots (2),$$

$$\Sigma \alpha^4 = 18s^4 + 12s^2t^2 + 2t^4 \dots (3), \text{ and } \Sigma \alpha^5 = -30s^5 + 20s^3t^2 + 10st^4 \dots (4).$$

$$(1) \text{ multiplied by (4) gives } \Sigma \alpha^2 \Sigma \alpha^5 = -180s^7 + 60s^5t^2 + 100s^3t^4 + 20st^6 \dots (5).$$

$$(2) \text{ multiplied by (3) gives } \Sigma \alpha^3 \Sigma \alpha^4 = -108s^7 + 36s^5t^2 + 60s^3t^4 + 12st^6 \dots (6).$$

The second member of (5) multiplied by 3 equals the second member of (6) multiplied by 5.

$$\therefore 3\Sigma \alpha^2 \Sigma \alpha^5 = 5\Sigma \alpha^3 \Sigma \alpha^4.$$

## II. Solution by ELMER SCHUYLER, Reading, Pa.

$$\text{By Newton's Theorem } \frac{xf'(x)}{f(x)} = n + \frac{\Sigma \alpha}{x} + \frac{\Sigma \alpha^2}{x^2} + \dots$$

In the present case  $f(x) \equiv x^3 + qx + r$ ,  $f'(x) = 3x^2 + q$ , and

$$\frac{xf'(x)}{f(x)} = 3 + \frac{0}{x} - \frac{2q}{x^2} - \frac{3r}{x^3} + \frac{2q^2}{x^4} + \frac{5qr}{x^5} \dots$$

$$\text{Consequently } \Sigma \alpha^2 = -2q, \quad \Sigma \alpha^5 = 5qr, \quad \Sigma \alpha^6 = -3r, \quad \Sigma \alpha^4 = 2q^2, \text{ and } 3\Sigma \alpha^2 \Sigma \alpha^5 = 5\Sigma \alpha^3 \Sigma \alpha^4 = -30q^2r.$$

## III. Solution by J. SCHEFFER, Hagerstown, Md.

Since  $\alpha + \beta + \gamma = 0$ ,  $\alpha\beta + \alpha\gamma + \beta\gamma = q$ ,  $\alpha\beta\gamma = -r$ , we find by squaring the first and applying the second,  $\Sigma \alpha^2 + 2q = 0$ , whence  $\Sigma \alpha^2 = -2q$ . By adding the three identities  $\alpha^3 + q\alpha + r = 0$ ,  $\beta^3 + q\beta + r = 0$ ,  $\gamma^3 + q\gamma + r = 0$ , we get  $\Sigma \alpha^3 + q\Sigma \alpha + 3r = 0$ , or since  $\Sigma \alpha = 0$ ,  $\Sigma \alpha^3 = -3r$ .

By adding the three identities  $\alpha^4 + q\alpha^2 + r\alpha = 0$ ,  $\beta^4 + q\beta^2 + r\beta = 0$ ,  $\gamma^4 + q\gamma^2 + r\gamma = 0$ , we get  $\Sigma \alpha^4 + q\Sigma \alpha^2 + r\Sigma \alpha = 0$ , whence  $\Sigma \alpha^4 = 2q^2$ .

By adding  $\alpha^5 + q\alpha^3 + r\alpha^2 = 0$ ,  $\beta^5 + q\beta^3 + r\beta^2 = 0$ ,  $\gamma^5 + q\gamma^3 + r\gamma^2 = 0$ , we get  $\Sigma \alpha^6 + q\Sigma \alpha^3 + r\Sigma \alpha^2 = 0$ , whence  $\Sigma \alpha^6 = 5qr$ .

$\therefore 3\Sigma \alpha^2 \cdot \Sigma \alpha^5 = -30q^2r$ ,  $5\Sigma \alpha^3 \cdot \Sigma \alpha^4 = -30q^2r$ , which proves the proposition.

## IV. Solution by F. D. POSEY, A. B., San Mateo, Cal.

In the following, since  $\Sigma \alpha = 0$  the terms containing  $\Sigma \alpha$  vanish.

$$\Sigma \alpha^2 = (\Sigma \alpha)^2 - 2\Sigma \alpha \beta = -2q.$$

$$\Sigma \alpha^3 = (\Sigma \alpha^2)(\Sigma \alpha) - \Sigma \alpha^2 \beta = -(\Sigma \alpha \beta)(\Sigma \alpha) + 3\alpha \beta \gamma = -3r.$$

$$\begin{aligned}\Sigma a^4 &= (\Sigma a^3)(\Sigma a) - \Sigma a^3 \beta = -(\Sigma a^3)(\Sigma a \beta) + \Sigma a^2 \beta \gamma = -(\Sigma a^2)(\Sigma a \beta) + (\Sigma a) a \beta \gamma \\ &= -(\Sigma a^2)(\Sigma a \beta) = 2q^2.\end{aligned}$$

$$\Sigma a^5 = (\Sigma a^4)(\Sigma a) - \Sigma a^4 \beta = -(\Sigma a^3)(\Sigma a \beta) + \Sigma a^3 \beta \gamma = -(\Sigma a^3)(\Sigma a \beta) + (\Sigma a^2) a \beta \gamma = 5r \eta.$$

$$\therefore 3\Sigma a^2 \Sigma a^5 = 5\Sigma a^3 \Sigma a^4.$$

\* \* \* Solved by G. B. M. Zerr and Elmer Schuyler by computing the values of  $\Sigma a^2$ ,  $\Sigma a^3$ ,  $\Sigma a^4$ ,  $\Sigma a^5$  by means of the symmetric functions of the roots.

Solved by L. E. Newcomb by actual determination of the roots  $\alpha$ ,  $\beta$ ,  $\gamma$ , of the given equation.

Also solved by G. W. Greenwood.

### GEOMETRY.

226. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

The triangles  $ABC$ ,  $A'B'C'$  are in plane perspective, and the corresponding sides  $BC$ ,  $B'C'$ , ..., cut in  $P$ ,  $Q$ ,  $R$ , respectively.  $AA'$ , ..., cut the line  $PQR$  in  $P'Q'R'$ , respectively. Show that  $(PP', QQ', RR')$  is an involution range.

Solution by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

$B(PP', QR) = B(P'P, Q'R')$  since they both equal the same range  $(B'P, DC)$  [ $D$  being the intersection of  $C'B'$  and  $AB$ ].

Therefore  $(PP', QR)$  and  $(P'P, Q'R')$  are equicross and  $(PP', QQ', RR')$  is an involution range by the theorem that if  $(AA', BC)$ ,  $(A'A, B'C')$  are equicross the range  $(AA', BB', CC')$  will be an involution. [Lachlan's *Modern Pure Geometry*, page 272, Art. 426].

229. Proposed by F. D. POSEY, A.B., San Mateo, Cal., and G. W. GREENWOOD, M.A. (Oxon), Lebanon, Ill.

The solutions of problem 219 in the April number, "devise a simple geometric solution of the general quadratic equation," give the roots when they are *real*. Required a construction for the roots when they are *complex*.

Solution by F. D. POSEY, A. B., San Mateo, Cal.

Dr. L. E. Dickson reports a solution on page 93 of the April issue of the MONTHLY which holds when the roots are *real*.

When the circle on  $AB$  does not cut  $Ox$  the roots are *complex*. From the center of the circle  $AB$  let fall a perpendicular upon  $Ox$  cutting the circle at  $C$  and  $Ox$  at  $P$ . Produce this line to  $D$  making  $CPD = q + 1$ . On  $CD$  as diameter describe a circle cutting  $Ox$  at  $M$  and  $N$ . Then the roots of the equation are  $OP + PM\sqrt{-1}$  and  $OP - PM\sqrt{-1}$ .

Proof.  $OP$  computed in terms of  $p$  and  $q$  is found to be

$$\frac{q+1 - \sqrt{[p^2 + (q-1)^2]}}{2}.$$